Sets, Categories and Topoi: approaches to ontology in Badiou's later work

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Abstract

Alain Badiou declared in his 1988 book *Being and Event* that "mathematics is ontology". His 2006 follow-up work, *Logics of Worlds*, remains committed to that declaration but deploys a shift in the type of mathematics at stake. Specifically, while *Being and Event* is primarily concerned with drawing out the ontological lessons of set theory, *Logics of Worlds* focuses on the theory of categories and topoi, a more recent mathematical innovation that many believe offers an alternative and superior "foundation" to mathematics than set theory.

Category theory arose in the latter half of the 20th century and has rapidly expanded to become a *lingua franca* of modern mathematics, uniting disparate disciplines and revealing deep underlying connections between hitherto unrelated areas of mathematics.

In sharp contrast to set theory's emphasis on articulating the internal structure of mathematical entities, category theory strips out all interiority and reduces mathematical entities to point-like "objects" with no internal structure. The mathematics arises instead from the network of relations between these objects: "arrows" that go from one object to another.

Sets can be recast in categorical terms by treating sets as objects and functions between sets as arrows between objects. By generalising the resulting category we obtain a general class of "set-like" categories

known as topoi. The theory of topoi is the central focus of *Logics of Worlds*. Badiou argues that they offer a means of thinking the ontology of appearance stripped of any traditional phenomenological inclinations: beings appear in a world rather than to a subject. Ontology thus involves two aspects: an *onto*-logy that considers being-qua-being (set theory) and an onto-*logy* that considers beings-in-a-world (the theory of topoi).

This paper outlines the rise of category theory as an alternative "foundation" for mathematics to sets, offering an account of its basic notions and a sketch of some of the ways in which topos theory generalises traditional set theory. This is then related to the shift in Badiou's work from "*onto*-logy" to "onto-*logy*". I end by flagging up a few of problems and issues with this move: (i) does it go far enough? (ii) does it fully address the often voiced criticism that Badiou's system has a weak account of relationality? (iii) doesn't it imply a rethink of Badiou's controversial theory of the state?

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1 Introduction

Mathematics is ontology. This is the famous opening declaration of *Being and Event*, Alain Badiou's major work to date, and it serves remarkably well as a three-word summary of the entire book. But despite Badiou's argument that mathematics *in general* is the discourse that expresses being qua being, the book itself concentrates almost exclusively on a *specific region* of mathematics: the theory of pure sets, as developed by Georg Cantor, Gottlob Frege, Ernst Zermelo, John von Neumann, Kurt Gödel, Paul Cohen and others over a roughly hundred year period starting from the 1870s.

This focus should not come as a surprise to anyone versed in contemporary mathematics. For both the interest in pure sets and the development of their theory was to a large extent driven by the foundational role claimed for them with respect to mathematics in general. Their extreme internal simplicity notwithstanding, sets can be nested within each other and thereby used to model pretty much any other mathematical entity. In other words, sets can be deployed in an ontological role *within* mathematics; they are the "ontology of ontology", so to speak. From this perspective everything "is" a set, in that the domain of mathematical discourse can more or less be identified with the universe of all possible sets.

Mathematicians were attracted by such a homogenous and systematic foundation because it seemed to hold out the prospect of finally ironing out the paradoxes and puzzles that have plagued the subject for centuries. Or so it was thought. For the development of set theory in fact uncovered a host of strange and unexpected phenomena that dashed any hopes of securing mathematics on a stable foundational plateau. In particular, Gödel's famous First Incompleteness Theorem demonstrated that for any reasonable axiomatisation of set theory one could concoct statements that were formally "undecidable", in that they were neither provable nor refutable from the axioms. Cohen later proved that these undecidable statements included certain basic questions about infinite cardinals, such as the Continuum Hypothesis that Cantor had fruitlessly spent his later years attempting to prove.

While the intellectual achievement of mathematical set theory is beyond doubt, the peculiarly ambiguous results of its programme could not help but give an impetus to those who had always been sceptical of set theory's foundational aspirations. This scepticism took many forms, but typically it admitted that sets could be used to model other any mathematical entity, but denied that these other entities actually *were* sets in any deeper ontological sense. It was rather a case of sets *simulating* other entities, or of other entities being "implemented" as sets, to use the mathematical jargon. In fact this attitude often went hand-in-hand with a more general scepticism towards ontological questions in general, and a concomitant "structuralist" emphasis on the operations and relations *between* mathematical entities rather than on the entities as such. As Quine put it: "There is no saying absolutely what the numbers are; there is only arithmetic."

Badiou is, of course, aware of these arguments and takes care in *Being and Event* to put forward a more sophisticated justification for his focus on set theory than merely asserting the ontological credentials of sets. In the book's Introduction he writes:

> I am not pretending in any way that the mathematical domains I mention [ie set theory and adjacent disciplines] are the most 'interesting' or significant in the current state of mathematics. That ontology has followed its course well beyond them is obvious. Nor am I saying that these domains are in a foundational position for mathematical discursivity, even if they generally occur at the beginning of every systematic treatise. To begin is not to found. My problem is not, as I have said, that of foundations, for that would be to advance within the internal architecture of ontology whereas my task is solely to indicate its site. However, what I do affirm is that historically these domains are *symptoms*, whose interpretation validates the thesis that mathematics is only assured of its truth insofar as it organises what, of being qua being, allows itself to be inscribed. (*B&E*, p14)

He immediately adds: "If other more active symptoms are interpreted then so much the better." This statement is significant precisely because another "more active symptom" was indeed rapidly emerging within mathematics during the latter half of the 20th century: category theory.

Categories were first developed in the early 1940s by Saunders Mac Lane and Samuel Eilenberg, two US-based mathematicians working on algebraic topology. Initially little more than a notational convenience, categories rapidly became formalised and started to spread to other areas of mathematics. Algebraic geometry in France was a notable "early adopter" of categories, especially under the influence of Alexander Grothendieck. It soon became clear that categories – along with associated paraphernalia such as functors, natural transformations and adjunctions – acted as a kind of universal mathematical language, a *lingua franca* illuminating hitherto unsuspected connections between different areas of mathematics.

We will examine categories in more detail later, but for now it suffices to think of a category as a collection of "objects" and "arrows" between them that satisfy certain laws. Typically – but not necessarily – the objects will be mathematical structures of a specified sort, and the arrows will be structure-preserving transformations between them. So for instance, we could form a category out of topological spaces and the continuous functions between them. Or we could collate together all groups and the homomorphisms between them. Or, to take a particularly simple but crucial example, we could consider the category of all sets and all functions between them.

The abstract elegance of categories, together with their seeming omnipresence and ability to connect disparate mathematical regions, soon raised the question of whether they could act as an alternative "foundation" to sets – and arguably a superior one, since categories arose directly out of and were immediately applicable to general mathematical practice, whereas set theory was a specialist mathematical subdiscipline whose techniques and insights were typically of little interest or use to mathematicians in other fields. The situation is further complicated by the fact that noted above: sets can themselves be considered as forming a category, a special kind of category known as a *topos*. It turns out that many of the key "foundational" properties of sets are also shared by topoi in general. This in turn suggests that the foundational role ascribed to sets is merely an arbitrary metaphysical prejudice on our part, since we could equally well ascribe the same role to some other collection of "set-like" entities inhabiting a topos.

Much of Badiou's work since *Being and Event* has involved getting to grips with these new "symptoms" and extending his meta-ontology to take account of them. An early effort to tackle category theory can be found in the essay 'Group, Category, Subject' collected in *Briefings on Existence*. But it is in 2006's *Logics of Worlds*, billed as the successor to *Being and Event*, that Badiou provides a thorough examination of topos theory. Roughly speaking, topoi play the equivalent role in *Logics of Worlds* to sets in *Being and Event*. In particular, Badiou argues that topoi encode the ontology of *appearance*. This is a strange kind of phenomenology where objects do not appear "to a subject", but rather "in a world". And the relationship between sets and topoi is thereby recast as the relationship between being-qua-being and being-there, or being-in-a-world.

In this essay I aim to do three things. First, I offer a basic account of sets, categories and topoi, basic enough to to be accessible to non-mathematicians, but detailed enough to equip them with the tools to follow the later arguments. Second, I use this account to shed a little light on how the systematic ontology of Badiou's later work interacts with that of *Being and Event*. Third, I raise some tentative questions about the compatibility between the two systems and ask whether the topos-based ontology of *Logics of Worlds* adequately replies to some of the criticisms of his earlier work.

2 Sets and functions

Georg Cantor, the founder of set theory, famously sought to define a set as any "grouping into a totality of quite distinct objects of our intuition or our thought". Badiou ruefully notes that the subsequent development of set theory served to undermine each of these concepts. Nevertheless, Cantor's definition remains a good place to start, since it captures the notion that a set is a *collection* of entities called its *elements*, and that furthermore it is *nothing more* than that collection: it has no further kind of structure, no order, spatiality or arithmetic. In particular we should note that two sets are deemed equal if and only if every element of one is an element of the other, and vice versa. It follows from this definition that the *empty set*, the set with no elements, is unique.

What exactly can we do with these minimally structured collections of elements? Very little until we notice that sets can themselves be elements of other sets. The set construction, in other words, can be nested – and it is this feature that imbues sets with the power to model far more complex mathematical entities. For instance, a set of two elements does not have any particular order imposed upon the pair: $\{a, b\}$ is the same thing as $\{b, a\}$. But what if we want an ordered pair $\langle a, b\rangle$, where the a comes first and the b comes second? In fact we can obtain such an entity using a trick discovered by the Polish mathematician Kazimierz Kuratowski: we define $\langle a, b\rangle$ to be the set $\{\{a\}, \{a, b\}\}$. The reader can check that $\langle a, b\rangle$ and $\langle b, a\rangle$ are distinct sets according to this definition, and that given an ordered pair, ie a set of the form $\{\{a\}, \{a, b\}\}$, we can reconstruct a and b respectively.

Given two sets a and b we say b is a *subset* of a if every element of b is also an element of a. It follows from this definition that every set is a subset of itself, and that every set has the empty set as a subset. But in general a set will have many other subsets apart from these two. We denote the set of all subsets of a by $\mathcal{P}a$ – this is called the *power set* of a. It plays a crucial role in Badiou's ontology: he calls $\mathcal{P}a$ the *state* of the situation a.

We can use these constructions of ordered pairs and subsets to create more complex structures. For instance, given two sets A and B we can consider the set of all ordered pairs $\langle a, b\rangle$ where a is an element of A and b is an element of B. This set is called the *Cartesian product* of A and B, denoted $A \times B$. A typical example of such a construction would be an infinite plane, which is precisely the set of all pairs of coordinates $\langle x, y \rangle$ where x and y are both drawn from the set $\mathbb{R}$ of real numbers. We can thus denote the plane as $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^2$.

What about a *relation* between two sets A and B? How do we model that using nothing but sets? For instance, A could be the set of all authors and B the set of all books: we might then wish to model the relation "a is the author of b". In fact this relation is simply a specified collection of ordered pairs $\langle a, b \rangle$, or equivalently, a specified subset of the product $A \times B$. Conversely, any subset of $A \times B$ defines a unique relation between elements of A and elements of B: a is related to b if and only if $\langle a, b \rangle$ is in the subset under consideration.

Mathematicians are particularly interested in relations between A and B that satisfy a further property: namely, that every element of A is related to exactly one element of B. Such a relation effectively assigns to each element a of A an element $f(a)$ of B. It is called a *function* from A to B and can be informally thought of as a "recipe" for turning an element of A into an element of B. For instance, the function $f(x) = x^2$ is a function from the real line $\mathbb{R}$ to itself. Considered as a relation it comprises the pairs $\langle x, y \rangle$ where $y = x^2$, ie a parabola in the plane.

3 From sets to categories

The mathematics of the above section is compressed but standard. Readers who are unsure of any details should consult any introductory text on set theory, where the same constructions will be spelled out at a more leisurely pace. The material is also covered in Appendix 2 of *Being and Event*, which includes some important comments by Badiou on the "structuralist illusion" in mathematics and the Heideggerean theme of the forgetting of being. For now it is enough to note that a *function* from one set to another can be treated *as* a set in its own right. Or, as Badiou puts it (discussing relations in general rather than functions):

> Who hasn't spoken, at one time or another, of the relation 'between' the elements of a multiple and therefore supposed that a difference in status opposed the elementary inertia of the multiple to its structuration? Who hasn't said 'take a set with a relation of order...', thus giving the impression that this relation was itself something completely different from a set? Each time, however, what is concealed behind this assumption of order is that being knows no other figure of presentation than that of the multiple, and that thus the relation, inasmuch as it is, must be as multiple as the multiple in which it operates. (*B&E*, p443)

We should also note in passing two fundamental properties of functions that play a crucial role in the transition to the category theoretic perspective on mathematics. First, functions can be *composed*. If f is a function from A to B and g is a function from B to C, we can define a new function that takes any a in A to $g(f(a))$ in C. This composite function from A to C is denoted $g \circ f$ (or simply gf). Moreover, composition when viewed as a binary operation on functions is *associative*, in that $h \circ (g \circ f) = (h \circ g) \circ f$ for any composable functions f, g and h.

Second, every set is equipped with a special function that sends each of its elements to itself. So for any set A, we can define a function e from A to A that "leaves everything as it is", ie sends each a in A to that a in A. This function is called the *identity* on A, and it acts as a neutral element in composition: $e \circ f = f$ for any function f to A and $g \circ e = g$ for any function g from A. From an ontological point of view this identity function is simply the "diagonal" subset of $A \times A$, ie the set of pairs $\langle a, a \rangle$ for each a in A.

It is these two properties of functions – associative composition and the existence of identies – that category theory takes as its starting point. Specifically, a *category* is defined to be a collection of "objects" and a collection of "arrows" between them. (We will ignore the troublesome question of what exactly a "collection" means in this context.) These arrows are subject to certain rules: (i) given any arrows f from A to B and g from B to C, there is a "composite" arrow $g \circ f$ from A to C; (ii) given f from A to B, g from B to C and h from C to D, then $h \circ (g \circ f) = (h \circ g) \circ f$; (iii) for any object A there is an "identity arrow" e from A to A such that $e \circ f = f$ and $g \circ e = g$ for all suitable f and g.

Clearly sets and the functions between them form a category, one typically denoted **Set**. And one can form plenty of other categories by considering sets equipped with a certain structure and functions that preserve that structure. For instance, we could take our objects to be groups (roughly speaking, sets equipped with a type of "multiplication" operation) and our arrows to be homomorphisms (functions between groups that "preserve" the multiplication). Or we could take our objects to be topological spaces and our arrows to be continuous functions between them.

However, the abstractive power of categories only comes into its own once we drop the assumption that objects somehow "ought" to be sets of some sort, and that arrows "ought" to be functions. For in general an object in a category has no internal structure whatsoever. It is simply a point, a momentary resting place where arrows arrive at or depart from. (In fact some presentations of category theory go even further, ditching objects altogether and defining categories solely in terms of their arrows.) And the arrows too have no internal mechanics: all that matters is their source, their target, and the composition relationships they enter into with other arrows.

The notion of *isomorphism* provides an indication of just how radical this abstraction is. An arrow f from A to B is called an isomorphism if it has an inverse, ie an arrow g from B to A such that $f \circ g$ and $g \circ f$ are both identity arrows. One can easily show that such an inverse, if it exists, is unique and is itself an isomorphism. Clearly all identity arrows are isomorphisms, and the composition of two isomorphisms yields a third. We say two objects A and B are *isomorphic* if there is an isomorphism between them. Isomorphism acts as an "equivalence relationship" among among objects: roughly speaking, if A and B are isomorphic then any arrows to or from A can be "translated" into equivalent arrows to or from B.

Now it can be shown that the language of category theory – the apparatus of arrows, composition, identities, sources and targets outlined above – is incapable of distinguishing between isomorphic objects inside a category. Insofar as an object can be specified using the terminology and techniques of category theory, it is specified "up to isomorphism" only. There could be several, or an infinite number of objects that fit the specification, but one can be assured that they will all be mutually isomorphic. From the point of view of categorical abstraction, isomorphic objects are "the same" – not quite identical, but indistinguishable and thereby admitting the highest degree of similarity appropriate to that abstraction.

The contrast here with sets could not be starker. The question of whether two sets are the same or not is governed by the axiom of extensionality: two sets are the same if and only if they have the same elements. Furthermore, the Axiom of Foundation ensures that the apparent circularity of this criterion can be dissipated. At the level of pure ontology, identity is an all-or-nothing affair. But the moment one steps onto phenomenological level described by categories, identity starts to blur: distinct-but-isomorphic objects can no longer be told apart.

Before we move on to consider topoi, we should briefly consider how this categorical phenomenology relates to Badiou's comments on the "structuralist illusion" mentioned above. Badiou's argument here proceeds on more-or-less Heideggerean lines: the naïve attitude of the working mathematician, who takes functions and relations at face value, represents a necessary abdication of the ontological vocation of mathematics: a "forgetting of being", a "technical domination", but one that is nevertheless an "imperative of reason", since mathematical practice would be impossible without it: "Actual mathematics is thus the metaphysics of the ontology that it is. It is, in its essence, *forgetting of itself*." (*B&E*, p446)

Badiou diverges from Heidegger at one point only: mathematics possesses internal resources to "forget the forgetting" and consequently should not be written off as a mere technical nihilism:

> Even if practical mathematics is necessarily carried out within the forgetting of itself – for this is the price of its victorious advance – the option of de-stratification is always available. . . In this sense, mathematical ontology is not technical, because the unveiling of the origin is not an unfathomable virtuality, it is rather an intrinsically available option, a permanent possibility. . . [Mathematics] is both the forgetting of itself and the critique of that forgetting. (*B&E*, p447)

But this account is unsatisfactory. It sets up an unacceptable island of privilege within mathematical praxis: aristocratic set theorists who apprehend the truth of the discipline versus plebian "working mathematicians" who labour under unfortunate structuralist delusions. More fundamentally, it puts ontological clarity and practical efficacy at loggerheads with each other. It concedes far too much to Heidegger by granting that the vast bulk of mathematical practice does indeed fall under "technical nihilism". Its response to Heidegger's charge sheet amounts to little more than special pleading for set theory.

Moreover, Badiou's own argument contains the germ of a more robust approach to these issues. For if mathematics can both forget being and forget that forgetting, then by the same token it can *forget the forgetting of the forgetting*. And in fact this fits far better to mathematical practice. Forgetting being corresponds to a naïve mathematical approach that does not broach ontological questions; forgetting the forgetting corresponds to grounding that naïve discourse on a rigorous ontological foundation; and forgetting the forgetting of the forgetting corresponds to understanding that the precise nature of that ontological foundation is arbitrary: what matters is that one *can* implement general mathematical entities as sets, not *how* that implementation takes place. From this perspective category theory is an extension and deepening of the ontological vocation of mathematics, rather than a reversion to illusion, or a sacrifice of theory on the altar of practice.

4 The categorical properties of sets

We mentioned above that sets and the functions between them form the category **Set**, and that the properties of objects in categories are cast in terms of the arrows between them rather than through reference to any internal structure of the object. It is worth seeing how this approach works in the case of **Set**, ie how we can characterise various properties of sets through "arrows only" definitions.

To begin with let's consider the concept of a *singleton set*, ie a set with exactly one element. Suppose $\{t\}$ is such a singleton and A is any other set. How many functions f are there from A to $\{t\}$? Exactly one, since $f(a)$ must be equal to t for every element a of A. In fact one can easily check that this property characterises singleton sets: if for every set A there is exactly one function from A to T, then T must be a singleton.

Now since we have characterised singletons purely in terms of functions, we can generalise this approach to any category. We say T is a *terminal object* of a category if and only if there is a unique arrow from A to T for every object A of the category. Not every category will have such terminal objects, but in those that do T will play a roughly analogous role to that played by singletons in **Set**.

We should note in passing that this arrows-based approach characterises singleton sets but does not and cannot specify *which* particular singleton we are talking about. More generally, we can show that any two terminal objects in any category are isomorphic, and that any object isomorphic to a terminal object is itself terminal. As mentioned above, category theory can only distinguish objects up to isomorphism. All singletons look "the same" as far as the pattern of arrows in and out of them are concerned. What the set is a singleton *of* is a question that cannot be framed, let alone answered, from a categorical perspective.

Terminal objects are just one example of a family of categorical constructions known as *limits*. Each of these defines a particular object and associated arrows "up to isomorphism" in terms of the uniqueness of arrows to or from that object. For instance, given any two objects A and B, we can define a *product* object $A \times B$ equipped with two projection maps from $A \times B$ to A and B respectively. Readers can consult the literature for the details of this construction; for now it will suffice to note that in **Set** the Cartesian product defined above, together with maps sending an ordered pair $\langle a, b \rangle$ to a and b respectively, acts as a product in the categorical sense too.

What about the *subset* construction? Is there a way of casting it in categorical language, or what amounts to the same thing, generalising the construction to that of a *subobject* in an arbitrary category? It turns out that the answer is yes – twice. There are two ways of describing subsets categorically, and the equivalence of the two turns out to be a fundamental property of the category **Set** – so fundamental that any other category sharing that property can be considered more or less "**Set**-like".

The first approach notes that any subset B of A comes equipped with a natural function j that takes any element of B to the same element considered as an element of A. This *inclusion* function, as it is known, has the property of being *monic*, or "left cancellable": if $j \circ g = j \circ h$, then $g = h$, for any pair of functions g and h from a common source to B. Moreover, any monic function to a set A defines a unique subset of A, namely the *image* of that function (ie those elements of A that are values of the function).

This approach generalises to an arbitrary category quite straightforwardly. Given any object A and a monic arrow j to A, we say j is a subobject of A. If j and k are two subobjects and h is an isomorphism such that $j = k \circ h$, we say j and k are isomorphic as subjobjects. One can easily check that in **Set**, a subjobject of a set A is simply an injective function to A, and this function is isomorphic as a subobject to the inclusion of its image in A.

The second approach starts from the observation that given any subset of A, every element of A is either in that subset or not. This means we can think of the subset as a function from A to 2, the two-element set $\{true, false\}$. This *characteristic* function, as it is known, returns the value *true* on elements of A that are in the subset and $false$ on those that are not. The subset can be reconstructed from the characteristic function as the *inverse image* of $\{true\}$, ie the set of all elements in A that are sent to *true* by the function. Conversely, every function from A to 2 defines a unique subset of A in this manner. Thus the subsets of A and the functions from A to 2 are essentially equivalent, in the sense that there is a natural way of converting one into the other and back again, from subset to characteristic and vice versa.

How do we perform this construction in an arbitrary category? We start with a special object Ω called the *subobject classifier* that plays the same role as 2 does in **Set**. We also specify an arrow t from T to Ω called the truth arrow, where T is a terminal object in our categor. This arrow to Ω plays the same role as the element *true* does in 2. Finally, we want to ensure that the arrows from an arbitrary object A to Ω correspond precisely to the subobjects of A (strictly speaking, to the isomorphic classes of subobjects of A). This amounts to stating that for any subobject j of A, there is a unique arrow χ from A to Ω such that j is the "pullback" of t along χ. A pullback is a special kind of limit, similar to the product construction mentioned earlier. Its precise details need not concern us here – the key point is that it plays a similar role to the inverse image of $\{true\}$ above.

To recap. In **Set** there is a precise and natural correspondence between the subsets of A and the functions from A to 2. Every subset A gives rise to such a function and vice versa. In a general category this translates into a correspondence (up to isomorphism) between subobjects of A and arrows from A to a special object Ω, known as the subobject classifier. Every subobject of A gives rise to an arrow from A to Ω and vice versa.

Of course there is no guarantee that a subobject classifier actually exists in any particular category. But if it does, we know that the objects of that category mimic those of **Set** with respect to the dialectic between wholes and parts. Instead of subsets we have subobjects – but those subobjects are well-behaved, in that they admit being indexed by maps to Ω.

5 Topoi and the logic of appearance

We are finally in a position to define topoi, the mathematical centrepiece of Badiou's *Logics of Worlds*. A topos is a category with two additional conditions: it is "Cartesian closed" and it has a subobject classifier. The category of sets and functions between them is a topos, with 2 playing the role of subobject classifier. But there are many other kinds of topoi that involve more complex subobject classifiers. These testify to a more complex logic of wholes and parts at work in these alternatives to our familiar set theoretical world.

First a brief word on what "Cartesian closed" means. A category is cartesian closed if it admits a basic "algebra" whereby objects have "products" and "exponents". We have already seen how in **Set** we can "multiply" two sets A and B to create their Cartesian product $A \times B$. Similarly we can define "A to the power B" to be the set of functions from B to A. A cartesian closed category is one where similar operations exist on its objects.

The more signifincant condition is the one about subobject classifiers. The existence of the subobject classifier – what Badiou calls a "transcendental" – imposes a certain set-like logic of wholes and parts upon the category. The precise nature of that logic is determined by the structure of the subobject classifier. This logic need not necessarily be classical, as it is in the case of **Set**. In fact the internal logic of a topos is typically constructive (or "intuitionist") rather than classical.

We have come a long way, so perhaps it is time to retrace our steps. We began with the doctrine of the pure multiple, or set theory. Starting from our ontological axioms we derived a host of secondary constructions on sets: subset, product, relation, function, characteristic. We then noted that the importance of these constructions lay not so much in what they were but in the grammar of relations they embodied. In particular, sets admit an *algebra* whereby they can be combined together through multiplication and exponentiation, together with a *dialectic* of wholes and parts. Neither this algebra nor this dialectic are built into the axiomatic foundation of sets; they arise out of those foundations. But any category that admits such an algebra and dialectic – any topos – can by virtue of this admission play the same foundational role as sets. Set theoretical ontology thus supplements itself in the course of its unfolding and points towards its own generalisation, the theory of topoi.

What then is the articulation between sets and topoi, between the "ontology of ontology" and the supplement it necessarily produces? What is at stake in this move from a *universe* of sets to a *multiverse* of topoi? – each with its own logic, its own dialectic of parts and wholes; the familiar **Set** being merely one of many possible worlds, many foundational choices.

Badiou deploys several terms here: logic, relation, appearance, phenomenology. Being is supplemented by being-there or being-in-a-world. Multiples do not just exist, they exist in a web of relations with other multiples. This web of relations constitutes a world where entities appear to one another. And the logic of that appearance is determined by the properties of that world. In particular the logic of wholes and parts – the dialectic alluded to above – is governed by that world's transcendental.

The study of the logic of appearance has is traditionally called phenomenology, and Badiou retains this term to describe the supplement to the pure ontology set forth in *Being and Event*. This is an appropriate choice in many ways, but we should note two crucial respects in which Badiou's "phenomenology" departs from that of the philosophical tradition.

First, phenomenology – certainly since Kant and arguably before – has typically involved appearance *to* a subject. Even Heidegger does not fundamentally depart from this conception: for him being is disclosed *to* a privileged entity that enquires: Dasein. But for Badiou appearance is appearance *in* a world. Entities appear without there being any-*one* to appear to. In this respect Badiou's phenomenology is a theory of objects without subjects that complements his earlier project of a subject without objects.

Second, phenomenology is *part* of ontology rather than being distinguished from it. Appearance is being-there and thus part of ontology proper.

It makes no sense to think of entities as they *are* on the one hand, and as they *seem to be* on the other. This envelopment of phenomenology within general ontology parallels the mathematicisation of logic, a revolution in thought that has unfolded over the past two hundred years. Recall that for Badiou ontology is mathematics – all of mathematics, not just set theory – and if phenomenology has been mathematicised, then it has been ontologised. Strictly speaking, the articulation is not between ontology and phenomenology as traditionally understood, but between *onto*-logy and onto-*logy*. Where we once had a metaphysical opposition, we now have a difference of emphasis.

6 The ontology of the multiple: decentred or dethroned?

We have described in some detail how Badiou's phenomenology of topoi supplements his pure ontology of sets. Yet this supplement comes at a price: it threatens to overwhelm and displace the origin that gave rise to it. One can see this tension at work both within Badiou's corpus and in mathematics more generally.

The contrast between set theoretical and categorical "foundations" of mathematics is a stark one. Arguably "contrast" is too mild a way of putting it: one would better speak of a *rivalry*. Thinking about mathematics in terms of categories involves a wrench away from familiar set theoretic assumptions, a wrench that requires considerable effort to master. We have seen how topoi act as the appropriate categorical generalisation of sets, and seen how this generalisation can bridge the gap between sets and categories, opening up a passage from the former to the latter. This is the strategy that Badiou adopts. By putting topoi at the centre of his later work, notably *Logics of Worlds*, Badiou can maintain his earlier pure ontology (characterised by classical decision and a refusal of relativistic sophistry) in an uneasy alliance with his later phenomenology (characterised by a ramification of possible worlds and a descriptive suspension of the decision between them). We see this tension in the following passage from Badiou's essay 'Being and Appearance', which appears in the English language collection *Theoretical Writings*:

> The theory of topoi is descriptive and not really axiomatic. The classical axioms of set theory fix the untotalisable universe of the thinking of the pure multiple. We could say that set theory constitutes an ontological decision. The theory of topoi defines, on the basis of an absolutely minimal concept of relation, the conditions under which it is acceptable to speak of a universe for thinking, and consequently to speak of the localisation of a situation of being. To borrow a Leibnizian metaphor: set theory is the fulminating presentation of a singular universe, in which what there is is thought, according to its pure "there is". The theory of topoi describes possible universes and their rules of possiblity. It is akin to the inspection of the possible universes which for Leibniz are contained in God's understanding. That is why it is not a mathematics of being, but a mathematical logic. (*TW*, p174)

So while Badiou admits a multiverse of possible universes, there is still a Platonic decision that accords an ontological privilege to one particular universe. We *might* be in any topos, but we *are* in **Set**. For many mathematicians this configuration remains a messy compromise. They urge that we go further and replace set theoretical foundations with ones based on categories and topoi. They are not satisfied with simply decentring the role of sets – they call for sets to be dethroned altogether. FW Lawvere, one of the founding figures of category theory, has long polemicised this point of view. At the very least it accords with the direction of mathematical practice – though whether Badiou would be moved by such empirical considerations remains somewhat questionable.

But this question is not merely one confined to mathematics or Badiou's ontological interpretation thereof. It impinges directly on at least two major debates surrounding Badiou's philosophical work. I want to end by briefly sketching these debates and the implications that categories and topoi have upon them. Hopefully this account will at the very least shed light upon the issues and perhaps point to directions for future research.

The first debate concerns the status of relationality in Badiou's work. As noted earlier, relations are a secondary concept in *Being and Event*, ones that arise out of the ontology of the pure multiple rather than being hard-coded into its axiomatic set-up. This has led to criticisms that Badiou's ontology remains impoverished and weak when it comes to grasping the relationality of being, since it does not accept relationality as a fundamental ontological category. In particular, it raises questions over whether Badiou's ontology is of any use in characterising the social and political world, where relations are generally considered paramount.

One could counter that the secondary status of relation in mathematical ontology has not had any limitative or deleterious effect on the thinking of relation within mathematics itself. On the contrary, mathematicians are constantly concerned with relations and have done so successfully for centuries. More than that – in recent decades mathematicians have discovered at least one hitherto unthought relation, that of *adjunction*, whose profundity and omnipresence in mathematics is only just beginning to be grasped.

But this rejoinder doesn't quite hit the mark. For the most creative thinking of relation, and the concept of adjunction in particular, has arisen out of category theory through the very gesture of bracketing off "foundational" ontological concerns. Category theory proceeds by ignoring the innards of mathematical entities and tackling them solely from the perspective of the arrows between them. The fact that such an approach leads to theoretical breakthroughs and innovations such as adjunction only fuels the suspicion that the pure ontology of set theory is at the very least dispensible. The possibility arises, therefore, that one can maintain Badiou's subordination of philosophy to its mathematical condition while radically transforming the content of the mathematical ontology that condition prescribes.

The second debate concerns perhaps the most controversial aspect of Badiou's work, his theory of the state. For the Badiou of *Being and Event* at any rate, the state is an ontological fixture. Its existence is guaranteed by an axiom – the power set axiom – in sharp contrast to the status of relation.

This is coupled to an explicit critique of the Marx-Engels theory of the state and an implicit critique of the revolutionary political practice it proposes.

This leads to a curious pessimism in Badiou's work. If the state is ontologically inscribed, it is in a sense immovable, and to that extent revolutions are doomed to failure. All they do is replace one kind of state with another, foreclosing the possibility of communism understood as a world without a state. Of course it is undeniable that the great revolutionary sequences of France, Russia and China have failed to dislodge the state, and much of Badiou's pessimism on this score takes its cue from this historical failure. But it is one thing to note the historical record and quite another to elevate it to an ontological principle. In practice Badiou's theory of the state finds itself allied to a sectarian and abstentionist political practice that ironically resembles that of certain Deleuze influenced autonomist currents that Badiou openly despises and condemns as petit bourgeois romanticism. Even worse, it risks endorsing a certain liberal common sense that views revolution as at best misguided idealism and at worst totaltiarian horror.

Yet are things as bleak as they seem? For Badiou's theory of the state is underpinned by an identification of the state of a situation a with the power set of that situation $\mathcal{P}a$ (the set of subsets of a particular set). And the machinery he deploys to analyse this relation – presentation, representation and the dialectic between them – is fundamentally tied to the specific configuration of that dialectic at work in classical Zermelo-Fraenkel set theory.

This picture is quite different in an arbitrary topos. Rather than working with sets and power sets, we have objects and power objects. And the relations between the two are governed by the transcendental, the subobject classifier, of that topos. At bottom the question of the state and situation is a question of the dialectic of parts and wholes. And as we have seen, topos theory admits far more variability on this score than classical set theory. So the transition from **Set** to an arbitrary topos involves a certain loosening of the relations between state and situation. This opens up the possiblity that we can hold fast to Badiou's central insight into the mathematical nature of ontology, hold fast to his fidelity to philosophy's mathematical condition, and even accept his identification of the state with power objects – while drawing radically different conclusions about the nature of the state and the viability of revolution.

In 1926 the mathematician David Hilbert issued a famous rejoinder to those who criticised Cantor's transfinite set theory as philosophically or theologically unacceptable: "No one shall expel us from the paradise that Cantor has created for us." One could say about category theory today that no one shall expel us from the paradise created by Eilenberg, Mac Lane, Lawvere and many others. Taking this injunction seriously, I submit, involves radicalising Badiou's ontology yet further and stripping it of its residual attachment to the classical set theory of Zermelo-Fraenkel – an attachment that despite its undoubted historical grandeur and brilliance can only serve as a conservative brake on thought today.

References

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